On the Definition of the Generalized Scattering Matrix of a Lossless Multiport

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***Abstract*—In this paper, we reconsider the question of the definition of the generalized scattering matrix (GSM) of a lossless device, and show the conditions that the GSM must satisfy in order to correctly represent a lossless device, independently of the method used for its calculation. Moreover, starting from circuit theory, possible choices are considered, and among them the one is examined that seems to be the most meaningful when dealing with modes below cutoff. When the circuit is lossless, in fact, the resulting GSM is *unitary*, even when modes below cutoff are taken as accessible. This property provides an immediate check of the correctness of the computational implementation of actual problems. Finally, a practical example of the usefulness of the conditions provided is shown.**

***Index Terms*—Scattering matrices, transmission line theory, waveguide discontinuities, waveguide junctions.**

I. Introduction

THE generalized scattering matrix (GSM) is certainly the most robust representation for modeling closely interacting discontinuities. In this case, several modes, mostly below cutoff, must be taken as accessible, as they contribute to the interaction between cascaded discontinuities [1]. The GSM is always stable, while other representations, such as the admittance/impedance or the transmission matrices, may become unstable when representing the interconnecting waveguides, as they contain the hyperbolic functions cosh or sinh, corresponding to evanescent modes. When modes below cutoff are accessible, however, the properties of the GSM defined as usual are quite different from those of the scattering matrix describing only modes above cutoff, say, $\mathbf{S}_c$. In particular, whoever has dealt with multimodal models of discontinuities has seen that their GSMs are not unitary, while the submatrix $\mathbf{S}_c$ is still unitary. Although this fact was noted many years ago [2], it is nevertheless very common to assume that losslessness implies the unitarity of the GSM (see [3]–[6]). A recent paper, concerning the use of the GSM in the context of the mode-matching technique [7], reconsidered the problem and showed that the GSM of a lossless and reciprocal junction must obey some constraints ([7, eqs. 52–53]). This is useful when checking the correctness of numerical implementation. The above criteria, however, only hold when the junction is abrupt, i.e., is located on a given transverse plane and, as such, suitable to direct application of mode matching. They cannot be applied if the discontinuity problem is of a different nature or if the scattering matrix is derived by a different technique. In this paper, instead, formulas are found that apply to any GSM, independently of the technique employed in the analysis. Moreover, it is considered the possibility of definitions of the GSM alternative to that currently adopted. In fact, it is common practice to define the scattering matrix of a linear junction as the one linking the amplitudes of the scattered modes to those of the incident ones, when the electrical ports are perfectly matched. This definition is physically meaningful when the electrical ports represent propagating modes, but it is rather obscure when ports represent evanescent modes.

For these modes, it is actually impossible to define a matched load. Nonetheless, because of the formal analogy with the case above cutoff, the amplitudes of the forward and backward attenuating terms are usually treated as incident and reflected waves, respectively. Hence, a mode below cutoff is considered as "matched" when there is only the forward attenuating term or, in other words, when the backward attenuating term vanishes. However, the behavior of modes above and below cutoff is rather different: a mode above cutoff carries maximum power when it is not reflected at all, while a mode below cutoff may carry power only via the interaction of the forward and backward attenuating terms.

It is, therefore, worthwhile to investigate the possibility of redefining the GSM in such a way that the discrepancies between modes above and below cutoff occurring in the traditional definition are overcome. Moreover, we explore a new definition of the GSM preserving the fundamental property of unitarity in order to retain an immediate and useful check on the formal correctness of the implementation.

In this paper, we show that the lack of unitarity of the GSM as currently defined only depends on an arbitrary normalization condition. Starting from circuit theory, it is possible to redefine the GSM in such a way that unitarity is preserved. Of course, such representation has the same validity as the classical one. It is emphasized that the approach is absolutely independent on the type of discontinuity and particular electromagnetic (EM) technique used for deriving the parameters and, therefore, we do not focus on any particular method. For a given junction, we assume that a matrix $\mathbf{Z}(\mathbf{Y})$ links the electric and magnetic field amplitudes of the accessible modes at the ports with respect to the accessible modes of the feeds. It is worth recalling that the accessible modes of a junction are the modes that cause interaction with adjacent discontinuities. Their number, mainly depending on the distance separating the discontinuities, is typically much smaller than that required to represent the unknown field at the interface. The representation of the latter requires, in fact, a number of local modes, that is in principle infinite.

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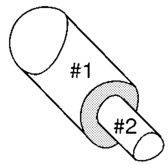

Fig. 1. Abrupt junction between two different waveguides of arbitrary cross section.

We will start considering the GSM of a length of waveguide interconnecting two successive discontinuities, which represents the fundamental block of any more complex circuit.

From the circuit point-of-view, this is equivalent to a set of parallel uncoupled transmission lines where the line corresponding to the fundamental mode has real characteristic impedance Z_0 and propagation constant β, whereas any remaining line has pure imaginary characteristic impedance jX_0 and real attenuation factor α.

II. Current Definition of $\mathbf{S}$ Matrix of a Length of Waveguide

The scattering matrix of a length of waveguide is currently taken as [8]

$$\begin{bmatrix} 0 & e^{-j\beta l} \\ e^{-j\beta l} & 0 \end{bmatrix} \text{ for a propagating mode} \tag{1}$$

$$\begin{bmatrix} 0 & e^{-\alpha l} \\ e^{-\alpha l} & 0 \end{bmatrix} \text{ for a nonpropagating mode} \tag{2}$$

where β is the propagation constant of the mode above cutoff and α is the attenuation of the mode below cutoff. Since there exists no physical definition based on wave amplitudes of the scattering matrix of a mode below cutoff, (2) is arrived at by setting $\alpha = -j\beta$ in (1). It is immediate to observe that the scattering matrix (SM) of a line above cutoff satisfies unitarity $\mathbf{S}^+\mathbf{S} = \mathbf{I}$, while the second ones gives

$$\mathbf{S}^+\mathbf{S} = \begin{bmatrix} e^{-2\alpha l} & 0 \\ 0 & e^{-2\alpha l} \end{bmatrix} \tag{3}$$

which is obviously not the unit matrix.

III. GSM of a Junction Modeled by Mode Matching

Let us consider now the abrupt junction between the two waveguides #1 and #2 (Fig. 1) as the one considered in a recent paper [7]. Under the mode-matching approach, the equivalent circuit is defined by [7, eqs. (6) and (14)], repeated here for clarity, representing the continuity of the electromagnetic tangential field, at the interface between the two waveguides #1 and #2

$$\mathbf{v}_1 = \mathbf{A}\mathbf{v}_2 \tag{4}$$

$$\mathbf{A}^t\mathbf{i}_1 = -\mathbf{i}_2 \tag{5}$$

where $\mathbf{v}_1 = [v_{11} \cdots v_{1N}]^t$, $\mathbf{v}_2 = [v_{21} \cdots v_{2M}]^t$, $\mathbf{i}_1 = [i_{11} \cdots i_{1N}]^t$ and $\mathbf{i}_2 = [i_{21} \cdots i_{2M}]^t$, respectively, represent the normalized modal voltages and currents on the left- and right-hand sides of the interface, N and M being the number of modes considered at each side. $\mathbf{A}$ is the coupling matrix.

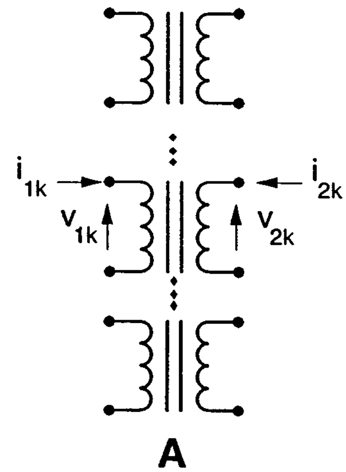

Fig. 2. Mode matching corresponds to an ideal transformer coupling modal lines.

The above equations correspond to a generalized ideal transformer, containing, in principle, an infinite number of elements, as the one depicted in Fig. 2. For such a transformer, it is immediate to show that $\mathbf{v}^t\mathbf{i} = 0$ as well as $\mathbf{v}^+\mathbf{i} = 0$, where $\mathbf{v}^t = [\mathbf{v}_1^t\mathbf{v}_2^t]$ and $\mathbf{i}^t = [\mathbf{i}_1^t\mathbf{i}_2^t]$, apex $^+$ denoting the adjoint matrix

$$\mathbf{v}_1^t\mathbf{i}_1 = \mathbf{v}_2^t\mathbf{A}^t\mathbf{i}_1 = -\mathbf{v}_2^t\mathbf{i}_2 \tag{6}$$

and

$$\mathbf{v}_1^+\mathbf{i}_1 = \mathbf{v}_2^+\mathbf{A}^+\mathbf{i}_1 = -\mathbf{v}_2^+\mathbf{i}_2 \text{ being } \mathbf{A} \text{ real.} \tag{7}$$

Now, expressing the above equations in terms of the scattering matrices of the circuit, we obtain

$$\mathbf{v}^t\mathbf{i} = [\mathbf{a}+\mathbf{b}]^t[\mathbf{a}-\mathbf{b}] = \mathbf{a}^t\left[\mathbf{U}+\mathbf{S}^t\right][\mathbf{U}-\mathbf{S}]\mathbf{a} = 0 \tag{8}$$

where $\mathbf{U}$ is the unit matrix. The above equation must be satisfied for an arbitrary excitation $\mathbf{a}$. This occurs only when

$$\left[\mathbf{U}+\mathbf{S}^t\right][\mathbf{U}-\mathbf{S}] = \mathbf{0}. \tag{9}$$

Now taking advantage from reciprocity, the above equation becomes

$$\mathbf{S}^t\mathbf{S} = \mathbf{U} \tag{10}$$

which is just [7, eq. (52)]. On the other hand, expressing (7) in terms of the scattering matrix, we obtain

$$\mathbf{v}^+\mathbf{i} = \mathbf{a}^+\left[\mathbf{U}+\mathbf{S}^+\right]\mathbf{Y}^{1/2*}\mathbf{Y}^{-1/2}[\mathbf{U}-\mathbf{S}]\mathbf{a} = 0 \tag{11}$$

which, because of the arbitrariness of $\mathbf{a}$, coincides with [7, eq. (53)]. The above equations, although useful to check the algebraic correctness of the GSM, are applicable to just abrupt junctions as they rely on the technique used (mode matching) to solve the EM problem. They do not hold in general, when the junction is of a more general kind or the circuit parameters are derived by a different method. They cannot be used for distance to check the formal correctness of a scattering matrix of a junction (not necessarily abrupt) analyzed either by the finite-element method (FEM) or finite-difference time-domain (FDTD) method, or variational methods.

IV. Properties of the GSM of an Arbitrary Lossless Circuit

In order to understand how the losslessness of a circuit reflects on the properties of $\mathbf{S}$, let us consider a N-port circuit fed by N transmission lines of characteristic impedance z_{0n}. Remember that each electrical line corresponds to an accessible modes, which may be either above or below cutoff. The circuit is characterized by the impedance (admittance) matrix $\mathbf{Z}(\mathbf{Y})$ linking the amplitudes of the modal electric fields $\mathbf{V}$ to those of the modal magnetic fields $\mathbf{I}$. The definition of the scattering matrix requires introducing the normalized voltages and currents $\mathbf{v}$ and $\mathbf{i}$ given by

$$\begin{aligned}\mathbf{v} &= \zeta^{-1/2}\mathbf{V}\\ \mathbf{i} &= \zeta^{1/2}\mathbf{I}\end{aligned} \tag{12}$$

where $V_n = V_n^+ + V_n^-$ and $I_n = I_n^+ + I_n^-$, $V_n^\pm$ are the amplitudes of the transverse electric field of the forward and backward (traveling or attenuating) modal wave, respectively, $I_n^\pm$ those of the corresponding transverse magnetic field. ζ is a diagonal matrix whose element ζ_i is the impedance normalizing the ith port of the circuit. The scattering matrix $\mathbf{S}$ is formally defined [9] as the matrix linking vectors $\mathbf{a}$ and $\mathbf{b}$, which, in turn, are related to the normalized modal voltages and currents as follows:

$$\begin{aligned}\mathbf{a} &= \frac{1}{2}(\mathbf{v}+\mathbf{i})\\ \mathbf{b} &= \frac{1}{2}(\mathbf{v}-\mathbf{i}).\end{aligned} \tag{13}$$

Therefore,

$$\mathbf{b} = \mathbf{S}\mathbf{a} = (\mathbf{z}-\mathbf{U})(\mathbf{z}+\mathbf{U})^{-1}\mathbf{a} \tag{14}$$

where $\mathbf{z}^{1/2}\mathbf{Z}^{-1/2}$ is the normalized impedance matrix. According to (13), the coefficients a_n and b_n pertaining to the nth feeding line are given by

$$\begin{aligned}a_n &= \frac{1}{2\sqrt{\zeta_n}}\left[V_n^+\left(1+\frac{\zeta_n}{z_{0n}}\right)+V_n^-\left(1-\frac{\zeta_n}{z_{0n}}\right)\right]\\ b_n &= \frac{1}{2\sqrt{\zeta_n}}\left[V_n^+\left(1-\frac{\zeta_n}{z_{0n}}\right)+V_n^-\left(1+\frac{\zeta_n}{z_{0n}}\right)\right].\end{aligned} \tag{15}$$

The expressions for a_n and b_n depend on the choice of the normalization impedance ζ_n. The most natural choice, indeed the one universally employed, is

$$\zeta_n = z_{\mathrm{on}}. \tag{16}$$

If the nth mode is *above cutoff*, it is immediate to note that, being z_{0n} real, a_n and b_n are proportional to the amplitudes of the forward and backward propagating waves, V_n^+ and V_n^-, respectively. In addition, the normalized voltages and currents satisfy the same power normalization as the unnormalized ones as follows:

$$P_n = V_n^* I_n = v_n^* i_n \tag{17}$$

if the mode is *below cutoff* a_n and b_n still represent the amplitudes of a forward and backward attenuating modes. Unfortunately, the normalized voltages and currents v_n, i_n do not satisfy the same power normalizations as the unnormalized ones V_n, I_n. We have, in fact,

$$P_n = V_n^* I_n \neq v_n^* i_n = V_n^* \frac{\zeta_n^{1/2}}{(\zeta_n^{1/2})^*} I_n. \tag{18}$$

As an immediate consequence, the scattering matrix of a lossless circuit is no longer unitary, for

$$\begin{aligned}\mathrm{Re}\left[\mathbf{V}^+\mathbf{I}\right] &= 0\\ &= \mathrm{Re}\left[(\mathbf{a}+\mathbf{b})^+(\zeta^{1/2})^*(\zeta^{-1/2})(\mathbf{a}-\mathbf{b})\right]\\ &\neq |\mathbf{a}|^2 - |\mathbf{b}|^2.\end{aligned} \tag{19}$$

It is apparent from the latter that losslessness implies unitarity of the scattering matrix only if the normalization impedances are real. For modes below cutoff, this requirement is in contrast to the normalization above.

Note also that the first line of (19) must be satisfied whatever the situation and the method is used to compute the GSM. Rewriting (19) in terms of the scattering matrix, the following equation is obtained:

$$\mathrm{Re}\left[\mathbf{a}^+\mathbf{C}\mathbf{a}\right] = 0 \tag{20}$$

where

$$\mathbf{C} = (\mathbf{U}+\mathbf{S})^+(\zeta^{1/2})^*(\zeta^{-1/2})(\mathbf{U}-\mathbf{S}). \tag{21}$$

In order to satisfy the above equation, $\mathbf{C}$ must have the following form:

$$\begin{aligned}\mathrm{Re}[C_{ii}] &= 0\\ \mathrm{Re}[C_{ij}] &= -\mathrm{Re}[C_{ji}]\\ \mathrm{Im}[C_{ij}] &= \mathrm{Im}[C_{ji}].\end{aligned} \tag{22}$$

The latter provides indeed the *most general criterion* to check the losslessness of the circuit, when its GSM is given. It deserves to be emphasized how fundamental it is to know the normalization conditions under which the GSM has been computed. This aspect is particularly important when the GSM must be built on the basis of a commercial software, for instance, by projecting the field on a given section on the accessible modes of the waveguide. Note also that the above criterion is much more useful than just considering the unitarity of the submatrix corresponding to modes above cutoff, as the latter check does not involve modes below cutoff.

Finally, it is noted that an alternative set of equations representing the losslessness of a circuit can be derived by the property of its impedance matrix $\mathbf{Z}$ to be imaginary. In fact, by expressing $\mathbf{Z}$ in terms of $\mathbf{S}$, we obtain

$$\mathrm{Re}\left[\zeta^{1/2}(\mathbf{U}+\mathbf{S})(\mathbf{U}-\mathbf{S})^{-1}\zeta^{1/2}\right] = 0. \tag{23}$$

It is possible to show that (20) and (23) are equivalent since the first one can be derived from the second and vice versa, as can be verified by direct substitution.

V. Alternative Definition of the GSM

It has been shown that the lack of unitarity of the GSM involving modes below cutoff is essentially due to the normalization adopted. On the other hand, nothing prevents one from choosing a different normalization, e.g., by setting

$$\zeta_n = |z_{\mathrm{on}}|. \tag{24}$$

For modes above cutoff, this position gives the same results as the traditional one, as can be immediately checked. For modes below cutoff, the relationship linking a_n and b_n to the amplitudes of the forward and backward attenuating modes is slightly more complicated than the standard one as follows:

$$\begin{aligned} a_n &= \frac{1}{2\sqrt{|z_{0n}|}}\left[V_n^+\left(1+\frac{1}{j\mathrm{sign}(z_{0n})}\right)+V_n^-\left(1-\frac{1}{j\mathrm{sign}(z_{0n})}\right)\right]\\ b_n &= \frac{1}{2\sqrt{|z_{0n}|}}\left[V_n^+\left(1-\frac{1}{j\mathrm{sign}(z_{0n})}\right)+V_n^-\left(1+\frac{1}{j\mathrm{sign}(z_{0n})}\right)\right]. \end{aligned} \tag{25}$$

Note that for modes below cutoff, a_n and b_n are no longer proportional to the amplitudes of the forward and backward attenuating modes V_n^+ and V_n^- (25). The scattering parameter $s_{ik} = b_i/a_k$ is now computed when the remaining ports are terminated on loads defined by the equation $a_n = 0$, $n \neq k$ or in terms of the forward and backward waves

$$V_n^+\big(1 - j\mathrm{sign}(z_{0n})\big) = -V_n^-\big(1 + j\mathrm{sign}(z_{0n})\big). \tag{26}$$

Choice (24) is legitimate and it perfectly matches all the requirements on the scattering matrix. In addition, the unitarity is preserved in the lossless case and, for a mode above cutoff, the interpretation of a_n and b_n as amplitudes of traveling waves is maintained.

Let us now consider how the above formalism applies to the case of a line below cutoff of characteristic impedance jX_0 normalized with respect to an arbitrary real impedance with real X_0 and attenuation $\xi = \alpha l$.

Starting from the well-known form of the Z-matrix of a length of the line

$$\mathbf{Z} = jX_0 \begin{bmatrix} \coth\xi & 1/\sinh\xi \\ 1/\sinh\xi & \coth\xi \end{bmatrix}. \tag{27}$$

Upon application of (14), we obtain

$$\mathbf{S} = \frac{1}{1 - X_0^2 + 2jX_0\coth\xi}\begin{bmatrix} -(1+X_0^2) & \dfrac{2jX_0}{\sinh\xi} \\ \dfrac{2jX_0}{\sinh\xi} & -(1+X_0^2) \end{bmatrix}. \tag{28}$$

We note that: 1) this matrix is unitary; 2) S_{11} and S_{22} do not vanish, in accordance with the fact that a line below cutoff reflects power; 3) as ξ tends to infinity, $|S_{11}|$ and $|S_{22}|$ tend to one, whereas S_{12} tends to zero; and 4) as ξ tends to zero, $|S_{11}|$ and $|S_{22}|$ tend to zero, whereas S_{12} tends to one when normalization (24) is assumed in the limiting case of a mode above cutoff, we have $X_0 = -j$, $\xi = j\beta l$ and the standard form (1) is recovered. In such case, the normalized current impedance of all lines below cutoff results to be $\mathrm{sign}(X_0)j$, while that of the lines above cutoff is one, according to standard convention. In that case, the 2 × 2 block corresponding to the fundamental mode takes the form

$$\mathbf{d}_1 = \begin{bmatrix} 0 & e^{-j\beta l} \\ e^{-j\beta l} & 0 \end{bmatrix} \tag{29}$$

whereas for a mode below cutoff

$$\mathbf{d}_k = \begin{bmatrix} j\mathrm{sign}(X_{0k})\tanh\xi_k & 1/\cosh\xi_k \\ 1/\cosh\xi_k & j\mathrm{sign}(X_{0k})\tanh\xi_k \end{bmatrix}, \quad k = 2, 3 \cdots. \tag{30}$$

The overall GSM of a length of waveguide with a single propagating mode takes the form

$$\mathbf{S} = \begin{bmatrix} \mathbf{0} & \mathbf{D} \\ \mathbf{D} & \mathbf{0} \end{bmatrix}$$

where

$$\mathbf{D} = \begin{bmatrix} \mathbf{d}_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{d}_2 & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{d}_N \end{bmatrix}. \tag{31}$$

Of course, the above normalization applies to the GSM of any linear device characterized by an impedance matrix $\mathbf{Z}$.

VI. Termination of the Alternative $\mathbf{S}$ Matrix at the Input and Output Ports

Since the input and output ports of the device are considered to be terminated by infinite lengths of waveguides, all ports corresponding to higher order modes have to be closed on their characteristic impedance, i.e., $j\,\mathrm{sign}(X_{0k})$ according to the normalization assumed in (24). Note that a load representing a pure reactive mode is modeled by a pure reactance $\pm j$, positive when the mode is inductive and negative when the mode is capacitive. Although both normalizations, the classical one and (24), are formally correct, the latter represents a mode below cutoff as a reactive load rather than a matched load. Therefore, the reduction of the GSM to the ordinary $2n_a \times 2n_a$ scattering matrix $\mathbf{s}$, where n_a is the number of modes *above* cutoff, takes place by using the standard port reduction formula

$$\mathbf{s} = \mathbf{S}_{aa} - \mathbf{S}_{ab}(\mathbf{J} - \mathbf{S}_{bb})^{-1}\mathbf{S}_{ba} \tag{32}$$

where $\mathbf{J}$ is a $(2n_b \times 2n_b)$ diagonal matrix, n_b being the number of *accessible* modes *below* cutoff, whose kth element is given by

$$[\mathbf{J}]_{kk} = [\mathbf{J}]_{k+n_b,\,k+n_b} = j\,\mathrm{sign}(X_{0k}), \qquad k = 1, n_b \tag{33}$$

block $\mathbf{S}_{ab}$ relates accessible modes of type a, above cutoff, to those of type b, below cutoff. Although the proposed definition of GSM is marginally more time consuming than the standard one, as it requires reduction, nonetheless it provides a very familiar criterion for checking numerical implementation through its unitarity. Conversely, it is apparent that one advantage of the classical definition is that no extra effort is required to reduce the GSM to that of the modes above cutoff.

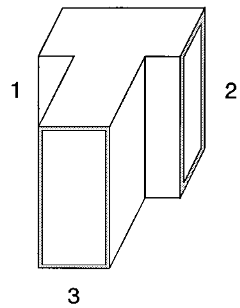

Fig. 3. E-plane T-junction.

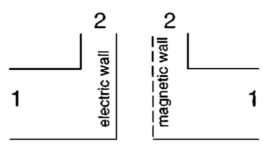

Fig. 4. E-plane section of the two E-plane 90° bends corresponding to the even and odd excitation at ports 1 and 2 of the T-junction.

VII. Usefulness of the Losslessness Conditions of the GSM

As previously observed, the main advantage of the new definition of the GSM is its unitarity since the latter constitutes an useful and immediate check to apply to numerical results. When using the standard definition of GSM, or any other correct definition, the losslessness of the circuit can be always checked by (22). When the junction is abrupt and mode matching is employed, one can use the check proposed in [7], which is, however, limited to this specific case. We note, however, that unitarity is much more immediate, and besides, most workers in the field do expect such a property from the scattering matrix of a lossless junction. Also, note that, depending upon the definition assumed, the GSM must satisfy one of the above conditions, that are, however, only *necessary*. In other words, their verification is not sufficient to check the correctness of the analysis.

Their application can be very useful for distance when dealing with symmetric components. In this case, the numerical effort is strongly reduced when symmetry is taken into account. Consider, for instance, the waveguide T-junction in the E-plane, shown in Fig. 3. Due to symmetry, the analysis of the junction is conveniently carried on by separately considering even and odd excitations at ports 1 and 2 in such a way that only two 90° bends have to be studied, as shown in Fig. 4. Suppose now to have computed the two scattering matrices $\mathbf{S}_e[(N_e+N)\times(N_e+N)]$ and $\mathbf{S}_o[(N_o+N)\times(N_o+N)]$, where N, N_e and N_o are the number of accessible modes at port 1, at port 3 in the even and odd cases. Both matrices $\mathbf{S}_e$ and $\mathbf{S}_o$ have the following form:

$$\mathbf{S}_g = \begin{bmatrix} \mathbf{S}_g^{11} & \mathbf{S}_g^{12} \\ \mathbf{S}_g^{21} & \mathbf{S}_g^{22} \end{bmatrix}, \qquad \text{where } g = e/o. \tag{34}$$

The global GSM $\mathbf{S}$ $[(N_e+N_o+2N)\times(N_e+N_o+2N)]$ of the junction is obtained combining the submatrices $\mathbf{S}_e$ and $\mathbf{S}_o$ as follows:

$$\mathbf{S} = \begin{bmatrix} \frac{1}{2}(\mathbf{S}_e^{11}+\mathbf{S}_o^{11}) & \frac{1}{2}(\mathbf{S}_e^{11}-\mathbf{S}_o^{11}) & \frac{1}{\sqrt{2}}\mathbf{S}_e^{12} & \frac{1}{\sqrt{2}}\mathbf{S}_o^{12} \\ \frac{1}{2}(\mathbf{S}_e^{11}-\mathbf{S}_o^{11}) & \frac{1}{2}(\mathbf{S}_e^{11}+\mathbf{S}_o^{11}) & \frac{1}{\sqrt{2}}\mathbf{S}_e^{12} & -\frac{1}{\sqrt{2}}\mathbf{S}_o^{12} \\ \frac{1}{\sqrt{2}}\mathbf{S}_e^{21} & \frac{1}{\sqrt{2}}\mathbf{S}_o^{21} & \mathbf{S}_e^{22} & \mathbf{0} \\ \frac{1}{\sqrt{2}}\mathbf{S}_e^{21} & -\frac{1}{\sqrt{2}}\mathbf{S}_o^{21} & \mathbf{0} & \mathbf{S}_o^{22} \end{bmatrix}. \tag{35}$$

Now, when building the GSM, suppose making the trivial, but very insidious mistake, consisting of setting $\mathbf{S}_{nm}^{14} = \mathbf{S}_{nm}^{24}\ \forall n, m$. It is immediate to observe that the block of the GSM relative to modes above cutoff $\mathbf{S}_c$ as normally defined, continues to satisfy unitarity. Therefore, an inspection of that property fails to detect the error. Also, [7, eqs. (52–53)] are not applicable, as the junction neither is abrupt nor is analyzed by mode matching. On the contrary, by checking either (22) or the unitarity, when the GSM is defined as proposed above, the mistake emerges immediately. This is just one of many simple and realistic examples showing the practical usefulness of the losslessness conditions.

VIII. Numerical Example

The classical GSM of the T-junction discussed above, computed at 8 GHz (the arms are WR90 waveguides), where we have taken as accessible modes LSE_{10} and LSE_{11} at the three ports, is given by

$$\mathbf{S} = \begin{bmatrix} -0.44-j0.32 & -0.22+j0.05 & -0.01-j0.57 & 0.11-j0.02 & -0.32+j0.08 & 0.31-j0.52 \\ -0.22+j0.05 & 0.09+j0.08 & 0.11-j0.02 & 0.06-j0.07 & 0.22+j0.02 & -0.30-j0.09 \\ -0.01-j0.57 & 0.11-j0.02 & -0.44-j0.32 & -0.22+j0.05 & -0.32+j0.08 & -0.31+j0.52 \\ 0.11-j0.02 & 0.06-j0.07 & -0.22+j0.05 & 0.09+j0.08 & 0.22+j0.02 & 0.30+j0.09 \\ -0.32+j0.08 & 0.22+j0.02 & -0.32+j0.08 & 0.22+j0.02 & 0.26+j0.11 & 0.00+j0.00 \\ 0.31-j0.52 & -0.30-j0.09 & -0.31+j0.52 & 0.30+j0.09 & 0.00+j0.00 & 0.07-j0.50 \end{bmatrix}. \tag{36}$$

The matrix $\mathbf{C}$, defined as above in (21), is given by

$$\mathbf{C} = \begin{bmatrix} 0.00+j0.48 & 0.17+j0.03 & 0.00+j1.08 & -0.10+j0.10 & 0.26+j0.02 & 0.05+j0.96 \\ -0.17+j0.03 & 0.00+j0.93 & 0.10+j0.10 & 0.00-j0.05 & 0.02-j0.09 & -0.26+j0.12 \\ 0.00+j1.08 & -0.10+j0.10 & 0.00+j0.48 & 0.17+j0.03 & 0.26+j0.02 & -0.05-j0.96 \\ 0.10+j0.10 & 0.00-j0.05 & -0.17+j0.03 & 0.00+j0.93 & 0.02-j0.09 & 0.26-j0.12 \\ -0.26+j0.02 & -0.02-j0.09 & -0.26+j0.02 & -0.02-j0.09 & 0.00+j0.82 & 0.00+j0.00 \\ -0.05+j0.96 & 0.26+j0.12 & 0.05-j0.96 & -0.26-j0.12 & 0.00+j0.00 & 0.00+j0.81 \end{bmatrix}. \tag{37}$$

As can be checked, the above matrix perfectly satisfies the requirements of (22). We do not report, for the sake of brevity, the GSM computed derived on the basis of the alternative normalization (24). In that case, it could be immediately checked that the GSM is unitary.

IX. CONCLUSIONS

In this paper, it has been shown that there are many possible correct definitions of the GSM of a junction depending upon the impedances chosen to normalize modal voltages and currents. A general criterion has been provided to check the GSM with respect to the losslessness of the device, independently of the junction itself and of the method used for its calculation. Also, it is possible to define the GSM in such a way that its unitarity is preserved when dealing with lossless devices. We have shown the usefulness by considering an example where unitarity permits to uncover an error that could easily occur in an incorrect implementation of the GSM of an E-plane T-junction.

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